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Stability and Control

Dimensional Derivatives

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Introduction

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- Complete Aircraft wing, tail and propulsion configuration, Mass Properties, including MOIs
- Non-Dimensional Derivatives (Roskam)
- Dimensional Derivatives (Etkin)
- Calculate System Matrix [A] and eigenvalues and eigenvectors
- Use results to determine stability (Etkin)

Reading:
Nicolai - CH 21, 22 & 23
Roskam – VI, CH 8 & 10

Other references:
MIL-STD-1797/MIL-F-8785 Flying Qualities of Piloted Aircraft
Airplane Flight Dynamics Part I (Roskam)

What are the requirements?

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Evaluate your aircraft for meeting the stability requirements
See SRD for values

- Flight Condition given:
  - Airspeed: M = ?
  - Altitude: ? ft.
  - Standard atmosphere
  - Configuration: ?
  - Fuel: ?%

- Longitudinal Stability:
  - $Cm_{CG\alpha} < 0$ at trim condition
  - Short period damping ratio: ?
  - Phugoid damping ratio: ?

- Directional Stability:
  - Dutch roll damping ratio: ?
  - Dutch roll undamped natural frequency: ?
  - Roll-mode time constant: ?
  - Spiral time to double amplitude: ?

Derivatives

- For General Equations of Unsteady Motions, reference Etkin, Chapter 4
- Assumptions
  - Aircraft configuration finalized
  - All mass properties are known, including MOI
  - Non-Dimensional Derivatives completed for flight condition analyzed
  - Aircraft is a rigid body
  - Symmetric aircraft across BL0, therefore $I_{xy}=I_{yz} = 0$
  - Axis of spinning rotors are fixed in the direction of the body axis and have constant angular speed
  - Assume a small disturbance
- Results in the simplified Linear Equations of Motion… on following slides
- But first: Convert non-dimensional derivatives to dimensional derivatives
  - Etkin, Tables 4.4 and 4.5 (also reference Table 4.1)
  - Takes flight path angle, reference areas/lengths, airspeed and altitude into account

Longitudinal Equation (Etkin Eq. 4.9,18)

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$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{x} + \Delta\mathbf{f}_c$$

$$\begin{bmatrix}\Delta\dot{u}\\ \dot{w}\\ \dot{q}\\ \Delta\dot{\theta}\end{bmatrix} = [4\ x\ 4\ Matrix]\begin{bmatrix}\Delta u\\ w\\ q\\ \Delta\theta\end{bmatrix} + \begin{bmatrix}\frac{\Delta X_c}{m}\\ \frac{\Delta Z_c}{m-Z_{\dot{w}}}\\ \frac{\Delta M_c}{I_y} + \frac{M_{\dot{w}}}{I_y}\frac{\Delta Z_c}{m-Z_{\dot{w}}}\\ 0\end{bmatrix}$$

Control forces, assume 0 for purposes of 460B

$\frac{X_u}{m}$	$\frac{X_w}{m}$	0	$-g\ \cos\theta_0$
$\frac{Z_u}{m - Z_{\dot{w}}}$	$\frac{Z_w}{m - Z_{\dot{w}}}$	$\frac{Z_q + mu_0}{m - Z_{\dot{w}}}$	$\frac{-mg\sin\theta_0}{m - Z_{\dot{w}}}$
$\frac{1}{I_y}\left[M_u + \frac{M_{\dot{w}}Z_u}{m - Z_{\dot{w}}}\right]$	$\frac{1}{I_y}\left[M_w + \frac{M_{\dot{w}}Z_w}{m - Z_{\dot{w}}}\right]$	$\frac{1}{I_y}\left[M_q + \frac{M_{\dot{w}}(Z_q + mu_0)}{m - Z_{\dot{w}}}\right]$	$\frac{-M_{\dot{w}}mg\sin\theta_0}{I_y(m - Z_{\dot{w}})}$
0	0	1	0

Eigenvalue/Eigenvector Review

$$\dot{X} = \mathbf{A}\mathbf{x} + \Delta \mathbf{f}_c$$

Solutions of the above first order differential equations are in the following form:

$$\boldsymbol{x}(\boldsymbol{t}) = \boldsymbol{x}_0 \boldsymbol{e}^{-\lambda t}$$

Eigenvector → x_0

Eigenvalue → λ

$$\lambda x_0 = \mathbf{A} x_0$$

$$(A - \lambda I) x_0 = 0$$

$$det(A - \lambda I) = 0$$

Expansion of the above results in the Nth degree characteristic equations

Longitudinal Stability

MATLAB example (values from Etkin, 6.2,1)

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Hint: Type command "format long" to obtain more decimal places

Command Window

Enter A Matrix

```
>> A=[-.006868 .01395 0 -32.2;-.09055 -.3151 773.98 0; .0001187 -.001026 -.4285 0; 0 0 1 0]

A =

   -0.0069    0.0140         0  -32.2000
   -0.0906   -0.3151  773.9800         0
    0.0001   -0.0010   -0.4285         0
         0         0    1.0000         0
```

Calculate eigenvalues and eigenvectors of Matrix A

```
>> [eigvec, eigval]=eig(A)

eigvec =

   0.0211 + 0.0166i   0.0211 - 0.0166i  -0.9983             -0.9983
   0.9996             0.9996            -0.0573 + 0.0097i  -0.0573 - 0.0097i
  -0.0001 + 0.0011i  -0.0001 - 0.0011i  -0.0001 - 0.0000i  -0.0001 + 0.0000i
   0.0011 - 0.0004i   0.0011 + 0.0004i   0.0001 + 0.0021i   0.0001 - 0.0021i


eigval =

  -0.3719 + 0.8875i        0                  0                  0
        0            -0.3719 - 0.8875i        0                  0
        0                  0            -0.0033 + 0.0672i        0
        0                  0                  0            -0.0033 - 0.0672i
```

Expand polynomial to obtain Characteristic Equation Coefficients

```
>> poly(eig(A))

ans =

    1.0000    0.7505    0.9355    0.0095    0.0042

fx >>
```


Longitudinal Stability

MATLAB example (values from Etkin, 6.2,1)

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Mode 2 (short period mode)

$$\lambda_{3,4} = -.3719 \pm .8875i$$

```
eigval =

  -0.3719 + 0.8875i        0                      0                      0
        0              -0.3719 - 0.8875i          0                      0
        0                    0              -0.0033 + 0.0672i            0
        0                    0                      0              -0.0033 - 0.0672i

>> poly(eig(A))

ans =

    1.0000    0.7505    0.9355    0.0095    0.0042
```

Mode 1 (Phugoid mode)

$$\lambda_{1,2} = -.003289 \pm .06723i$$

$$\lambda^4 + .7505\lambda^3 + .9355\lambda^2 + .0095\lambda + .0042 = 0$$

Note: Characteristic Equation Roots contain real and imaginary roots
A negative real root means it is a stable mode

$$\lambda = n \pm \omega i$$

Quantitative Analysis

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- Period, $T = \dfrac{2\pi}{\omega}$
- Time to double (t_{double}) or time to half (t_{half}) $\quad t_{double} \text{ or } t_{half} = \dfrac{\ln(2)}{|\zeta|\omega_n}$
- Cycles to double (N_{double}) or cycles to half (N_{half}) $\quad N_{double} \text{ or } N_{half} = .110\dfrac{\sqrt{1-\zeta^2}}{|\zeta|}$

$$\lambda = n \pm \omega i$$

Mode 2 (short period mode) $\quad \lambda_{3,4} = -.3719 \pm .8875i$

Mode 1 (Phugoid mode) $\quad \lambda_{1,2} = -.003289 \pm .06723i$

$\omega_n = \sqrt{\omega^2 + n^2}$, undamped circular frequency

$\zeta = \dfrac{-n}{\omega_n}$, damping ratio

Lateral Equation (Etkin Eq. 4.9,19)

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$$\begin{bmatrix} \dot{v} \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \end{bmatrix} = [4\ x\ 4\ Matrix] \begin{bmatrix} v \\ p \\ r \\ \phi \end{bmatrix} + \begin{bmatrix} \frac{\Delta Y_c}{m} \\ \frac{\Delta L_c}{I'_x} + I'_{zx} N_c \\ I'_{zx}\Delta L_c + \frac{\Delta N_c}{I'_z} \\ 0 \end{bmatrix}$$

Control forces, assume 0 for purposes of 460B

$\frac{Y_v}{m}$	$\frac{Y_p}{m}$	$\frac{Y_r}{m} - u_0$	$g\ \cos\theta_0$
$\frac{L_v}{I'_x} + I'_{zx}N_v$	$\frac{L_p}{I'_x} + I'_{zx}N_p$	$\frac{L_r}{I'_x} + I'_{zx}N_r$	0
$I'_{zx}L_v + \frac{N_v}{I'_z}$	$I'_{zx}L_p + \frac{N_p}{I'_z}$	$I'_{zx}L_r + \frac{N_r}{I'_z}$	0
0	1	$\tan\theta_0$	0

Lateral Equation (Etkin Eq. 4.9,19), continued

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Where:

$$I'_x = \left(\frac{I_x I_z - I_{zx}^2}{I_z}\right)$$

$$I'_z = \left(\frac{I_x I_z - I_{zx}^2}{I_x}\right)$$

$$I'_{zx} = \left(\frac{I_{zx}}{I_x I_z - I_{xz}^2}\right)$$

$\frac{Y_v}{m}$	$\frac{Y_p}{m}$	$\frac{Y_r}{m} - u_0$	$g\ \cos\theta_0$
$\frac{L_v}{I'_x} + I'_{zx} N_v$	$\frac{L_p}{I'_x} + I'_{zx} N_p$	$\frac{L_r}{I'_x} + I'_{zx} N_r$	0
$I'_{zx} L_v + \frac{N_v}{I'_z}$	$I'_{zx} L_p + \frac{N_p}{I'_z}$	$I'_{zx} L_r + \frac{N_r}{I'_z}$	0
0	1	$\tan\theta_0$	0

MATLAB example (values from Etkin, 6.7,1)

Hint: Type command "format long" to obtain more decimal places

Enter B Matrix

```
B =

   -0.0558         0 -774.0000   32.2000
   -0.0039   -0.4342    0.4136         0
    0.0011   -0.0061   -0.1458         0
         0    1.0000         0         0
```

Calculate eigenvalues and eigenvectors of Matrix B

```
>> [eigvec, eigval]=eig(B)

eigvec =

  -1.0000             -1.0000             -0.9972              0.9821
   0.0019 - 0.0032i    0.0019 + 0.0032i   -0.0367             -0.0014
  -0.0001 + 0.0011i   -0.0001 - 0.0011i    0.0021              0.0078
  -0.0035 - 0.0019i   -0.0035 + 0.0019i    0.0652              0.1880


eigval =

  -0.0330 + 0.9465i        0                   0                   0
        0             -0.0330 - 0.9465i        0                   0
        0                  0             -0.5625                   0
        0                  0                   0             -0.0073
```

Expand polynomial to obtain Characteristic Equation Coefficients

```
>> poly(eig(B))

ans =

    1.0000    0.6358    0.9388    0.5114    0.0037
```

MATLAB example (values from Etkin, 6.2,1)

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Mode 3 (lateral oscillation or Dutch Roll)

$$\lambda_{3,4} = -.0330 \pm .9465i$$

```
eigval =

  -0.0330 + 0.9465i         0                  0          0
        0            -0.0330 - 0.9465i         0          0
        0                   0             -0.5625         0
        0                   0                  0    -0.0073

>> poly(eig(B))

ans =

    1.0000    0.6358    0.9388    0.5114    0.0037
```

Mode 1 (Spiral mode)

$$\lambda_1 = -.0073$$

Mode 2 (Rolling Convergence)

$$\lambda_2 = -.5625$$

$$\lambda^4 + .6358\lambda^3 + .9388\lambda^2 + .5114\lambda + .0037 = 0$$

Note: Characteristic Equation Roots contain real and imaginary roots
A negative real root means it is a stable mode

$$\lambda = n \pm \omega i$$

Lateral Stability

Results

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Etkin Table 4.2

	Cx	Cz	Cm
u_hat	Cx_u	Cz_u	Cm_u
α	Cx_α	Cz_α	Cm_α
q_hat	Cx_q	Cz_q	Cm_q
α_dot_hat	$Cx_{\alpha dot}$	$Cz_{\alpha dot}$	$Cm_{\alpha dot}$

	Cx	Cz	Cm
u_hat	-0.108	-0.106	0.1043
α	0.2193	-4.92	-1.023
q_hat	0	-5.921	-23.92
α_dot_hat	0	5.896	-6.314

Etkin Table 4.3

	Cy	Cl	Cn
β	Cy_β	Cl_β	Cn_β
p_hat	Cy_p	Cl_p	Cn_p
r_hat	Cy_r	Cl_r	Cn_r
β_dot_hat	$Cy_{\beta dot}$	$Cl_{\beta dot}$	$Cn_{\beta dot}$

	Cy	Cl	Cn
β	-0.8771	-0.2797	0.1946
p_hat	0	-0.3295	-0.04073
r_hat	0	0.304	-0.2737
β_dot_hat	0	0	0

Etkin Table 4.4

	X	Z	M
u	X_u	Z_u	M_u
w	X_w	Z_w	M_w
q	X_q	Z_q	M_q
w_dot	X_{w_dot}	Z_{w_dot}	M_{w_dot}

	X	Z	M
u	-1.346E+02	-1.776E+03	3.551E+03
w	2.734E+02	-6.133E+03	-3.483E+04
q	0.000E+00	-1.008E+05	-1.112E+07
w_dot	0.000E+00	1.296E+02	-3.791E+03

Etkin Table 4.5

	Y	L	N
v	Y_v	L_v	N_v
p	Y_p	L_p	N_p
r	Y_r	L_r	N_r

	Y	L	N
v	-1.093E+03	-6.824E+04	4.747E+04
p	0.000E+00	-7.866E+06	-9.723E+05
r	0.000E+00	7.257E+06	-6.534E+06

Side Bar: Routh's Criteria for Stability

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Use to check for instability

Start with the characteristic equation

$$A\lambda^4+B\lambda^3+C\lambda^2+D\lambda+E=0 \ (A>0)$$

$$R = D(BC - AD) - B^2E > 0$$

Routh's discriminant

<u>E = 0 and R = 0 are the boundaries between stability and instability</u>

Assume a stable aircraft, and you change a design parameter resulting in an instability, then the following conditions hold:

- If only E changes from + to -, then one real root changes from – to +, that is, one divergence appears in the solution.
- If only R changes from + to -, then one real part of one complex pair of roots changes from – to +, that is, one divergent oscillation appears in the solution.

Stability Results

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Moments of Inertia CGref		
Ixx	1.83E+07	slugs-ft^2
Iyy	3.31E+07	slugs-ft^2
Izz	4.97E+07	slugs-ft^2
Izx	-1.56E+06	slugs-ft^2
Ixx'	1.83E+07	slugs-ft^2
Izz'	4.96E+07	slugs-ft^2
Izx'	-1.72E-09	slugs-ft^2

A (Longitudinal Equations, Etkin 4.9,18)			
-6.8040E-03	1.3816E-02	0.0000E+00	-32.174
-0.0904	-0.3120	774.3	0
1.1512E-04	-1.0165E-03	-4.2462E-01	0
0	0	1	0

Characteristic Equation Coefficients				
A	B	C	D	E
1	0.750468	0.935494	0.009463	0.0041959

R	0.00419
E>0?	STABLE
R>0?	STABLE

B (Lateral Equations, Etkin 4.9,19)			
-5.526E-02	0.000E+00	-7.743E+02	32.174
-3.820E-03	-4.293E-01	4.089E-01	0
1.075E-03	-6.088E-03	-1.443E-01	0
0	1	0	0

Characteristic Equation Coefficients				
A	B	C	D	E
1	0.6358	0.9388	0.5114	0.003682

R	0.0422
E>0?	STABLE
R>0?	STABLE

Stability Results

- Cells in BLUE are the requirements to meet in the SRD

	Mode	λ	real n	imaginary ω	Stable/Unstable?	undamped circular frequency ω_n (rad/s)	damping ratio ζ	Period T (s)	time to double or half t (s)	Cycles to double or half N	Time Contant τ_R (s)
Longitudinal	**1 (Phugoid)**	1,2	-0.003289	0.06723	**STABLE**	0.0673	0.04886	93.5	210.7	2.25	
	2 (Short Period)	3,4	-0.3719	0.8875	**STABLE**	0.9623	0.38648	7.1	1.9	0.26	
Lateral	**1 (Spiral)**	1	-0.0072973		**STABLE**	0.0073	1.00000	n/a	95.0	n/a	
	2 (Rolling Convergence)	2	-0.56248		**STABLE**	0.5625	1.00000	n/a	1.2	n/a	1.8
	3 (Lateral Oscillation/Dutch Roll)	3,4	-0.033011	0.94655	**STABLE**	0.9471	0.03485	6.6	21.0	3.15	

Note: τ_R (time constant) = $-1/\lambda_2$

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Backup

Control Axis

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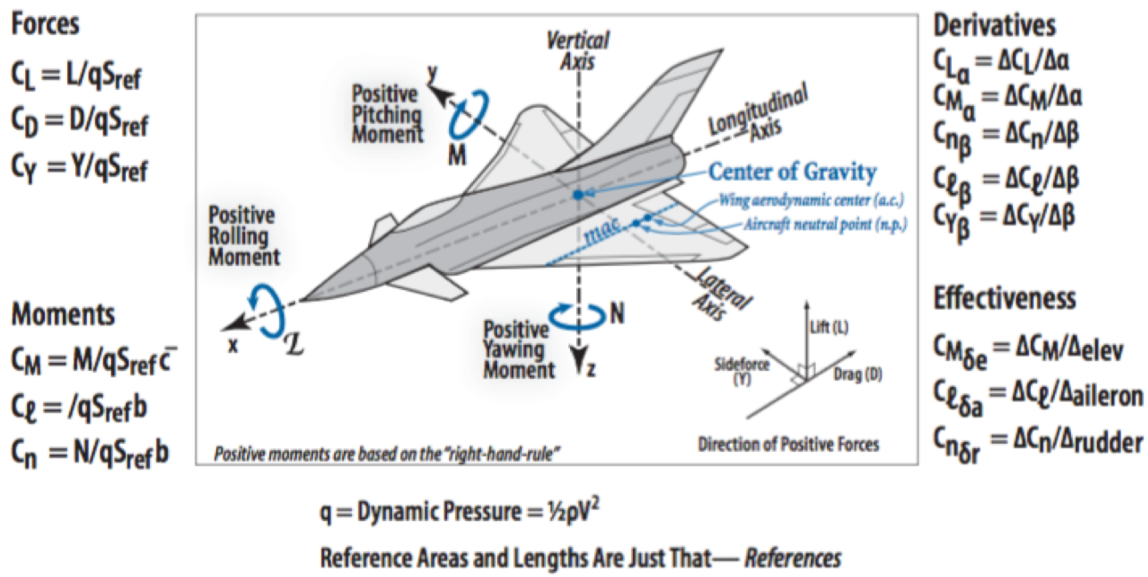

Figure 21.1 Major nondimensional aerodynamic parameters and sign convention. (Nicolai)

(Nicolai)

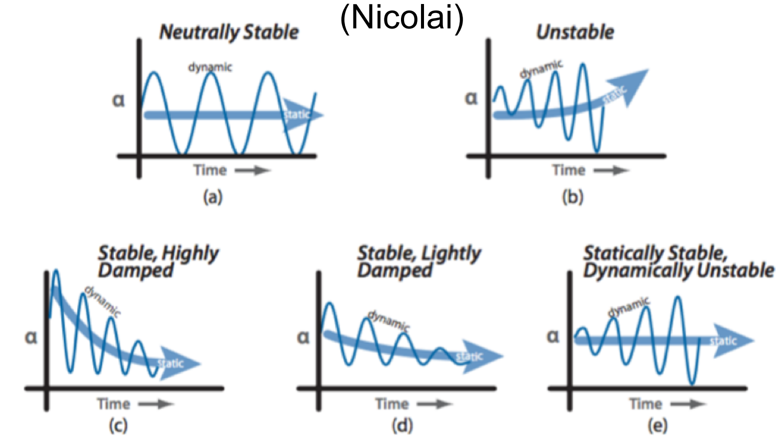

Figure 21.2 Static and dynamic stability about the pitch axis.

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